

COEFFICIENT ESTIMATES OF ANALYTIC ENDOMORPHISMS OF THE UNIT DISK FIXING A POINT WITH APPLICATIONS TO CONCAVE FUNCTIONS

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ABSTRACT. In this note, we discuss the coefficient regions of analytic self-maps of the unit disk with a prescribed fixed point. As an application, we solve the Fekete-Szegő problem for normalized concave functions with a prescribed pole in the unit disk.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in the complex plane \mathbb{C} . The class \mathcal{B}_p for $p \in \mathbb{D}$ will mean the set of holomorphic maps $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ satisfying $\varphi(p) = p$. In what follows, without loss of generality, we will always assume that $0 \leq p < 1$.

A function $\varphi \in \mathcal{B}_p$ can be expanded about the origin in the form

$$(1.1) \quad \varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots = \sum_{n=0}^{\infty} c_n z^n.$$

Note that $|c_n| \leq 1$ for each n . We define the coefficient body $\mathbf{X}_n(\mathcal{F})$ of order $n \geq 0$ for a class \mathcal{F} of analytic functions at the origin as the set

$$\{(c_0, c_1, \dots, c_n) \in \mathbb{C}^{n+1} : \varphi(z) = c_0 + c_1 z + \cdots + c_n z^n + O(z^{n+1}) \text{ for some } \varphi \in \mathcal{F}\}.$$

Note that $\pi_{m,n}(\mathbf{X}_n(\mathcal{F})) = \mathbf{X}_m(\mathcal{F})$ for $0 \leq m < n$, where $\pi_{m,n} : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$ is the projection $(c_0, c_1, \dots, c_n) \mapsto (c_0, c_1, \dots, c_m)$.

Obviously, $\mathbf{X}_0(\mathcal{B}_0) = \{0\}$ and $\mathbf{X}_1(\mathcal{B}_0) = \{(0, c) : |c| \leq 1\}$. In the present paper, we describe $\mathbf{X}_n(\mathcal{B}_p)$ for $n = 0, 1$ and $0 < p < 1$. Note that the authors describe $X_2(\mathcal{B}_p)$ in [11] to investigate the second Hankel determinant. In the following, it is convenient to put

$$P = p + \frac{1}{p} = \frac{1 + p^2}{p}.$$

Note that $P > 2$.

Theorem 1. *Let $p \in (0, 1)$.*

(i) $\mathbf{X}_0(\mathcal{B}_p) = \{c_0 \in \mathbb{C} : |c_0 - P^{-1}| \leq P^{-1}\}$. *For a function $\varphi(z) = c_0 + c_1 z + \cdots$ in \mathcal{B}_p , $c_0 \in \partial \mathbf{X}_0(\mathcal{B}_p)$ if and only if φ is an analytic automorphism of \mathbb{D} .*

(ii) $\mathbf{X}_1(\mathcal{B}_p) = \left\{ (c_0, c_1) \in \mathbb{C}^2 : |c_1 - (1 - Pc_0 + c_0^2)| \leq P \left[P^{-2} - |c_0 - P^{-1}|^2 \right] \right\}$. *In other words, a pair (c_0, c_1) of complex numbers is contained in $\mathbf{X}_1(\mathcal{B}_p)$ if and only if*

$$(1.2) \quad c_0 = P^{-1}(1 - \sigma_0) \quad \text{and} \quad c_1 = P^{-2}[1 + (P^2 - 2)\sigma_0 + \sigma_0^2] + P^{-1}(1 - |\sigma_0|^2)\sigma_1$$

2010 *Mathematics Subject Classification.* to be specified.

Key words and phrases. fixed point, concave functions, Dieudonne's lemma, variability region.

The first author was supported by Grant-in-Aid for JSPS Fellows No. 26 · 2855.

for some $\sigma_0, \sigma_1 \in \overline{\mathbb{D}}$.

Moreover, for a function $\varphi(z) = c_0 + c_1 z + \cdots$ in \mathcal{B}_p , $(c_0, c_1) \in \partial \mathbf{X}_1(\mathcal{B}_p)$ if and only if φ is either an analytic automorphism of \mathbb{D} or a Blaschke product of degree 2.

Our motivation of the present study comes from an intimate relation between \mathcal{B}_p and the class \mathcal{C}_{o_p} of concave functions f normalized by $f(0) = f'(0) - 1 = 0$ with a pole at p . Here, a meromorphic function f on \mathbb{D} is said to be *concave*, if it maps \mathbb{D} conformally onto a concave domain in the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; in other words, f is a univalent meromorphic function on \mathbb{D} such that $\mathbb{C} \setminus f(\mathbb{D})$ is convex. The class \mathcal{C}_{o_p} is intensively studied in recent years by Avkhadiiev, Bhowmik, Pommerenke, Wirths and others (see e.g. [1, 2, 3, 4, 5, 6]).

The following representation of concave functions in terms of functions in \mathcal{B}_p is contained in the first author's paper [10].

Theorem A. *Let $0 < p < 1$ and put $P = p + 1/p$. A meromorphic function f on \mathbb{D} is contained in the class \mathcal{C}_{o_p} if and only if there exists a function $\varphi \in \mathcal{B}_p$ such that*

$$(1.3) \quad f'(z) = (1 - Pz + z^2)^{-2} \exp \int_0^z \frac{-2\varphi(\zeta)}{1 - \zeta\varphi(\zeta)} d\zeta.$$

For a given function $f \in \mathcal{C}_{o_p}$ with the expansion

$$(1.4) \quad f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n=1}^{\infty} a_n z^n, \quad |z| < p,$$

we consider the Fekete-Szegő functional

$$\Lambda_{\mu}(f) = a_3 - \mu a_2^2$$

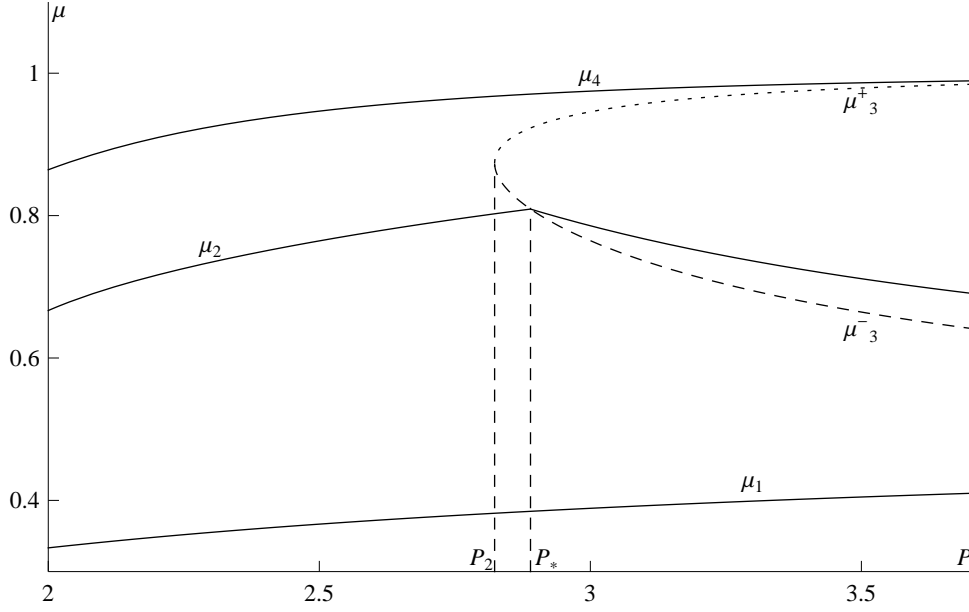
for a real number μ . For example, $\Lambda_1(f) = a_3 - a_2^2 = S_f(0)/6$, where $S_f = (f''/f')' - (f''/f')^2/2$ is the Schwarzian derivative of f . For some background of the Fekete-Szegő functional, the reader may refer to [7] and references therein. As an application of Theorem 1, we will prove the following.

Theorem 2. *Let $0 < p < 1$ and $\mu \in \mathbb{R}$ and put $P = p + 1/p$. Then the maximum $\Phi(\mu)$ of the Fekete-Szegő functional $|\Lambda_{\mu}(f)|$ over $f \in \mathcal{C}_{o_p}$ is given as follows:*

$$\Phi(\mu) = \begin{cases} (1 - \mu)P^2 - 1 & \text{if } \mu \leq \mu_1(P), \\ -\frac{1}{3}(P^3 - 2P + 3) + \frac{(P + 2)^2(2P - 1)^2}{12(P + 3\mu)} & \text{if } \mu_1(P) \leq \mu \leq \mu_2(P), \\ \Psi(P, \mu) & \text{if } \mu_2(P) \leq \mu \leq \mu_4(P), \\ (\mu - 1)P^2 + 1 & \text{if } \mu_4(P) \leq \mu. \end{cases}$$

Here,

$$\Psi(P, \mu) = \begin{cases} P^2 - 3 - \mu(P^2 - 4 + 4P^{-2}) & \text{if either } P_2 \leq P \leq P_*, \mu_3^-(P) \leq \mu \leq \mu_3^+(P) \\ \text{or } P_* \leq P, \mu_2(P) \leq \mu \leq \mu_3^+(P), \\ (1 - \mu)P(P^2 - 2) \sqrt{\frac{P^2 - 4\mu}{4\mu\{(1 - \mu)(P^2 - 1)^2 - 1\}}} & \text{otherwise,} \end{cases}$$

FIGURE 1. The graphs of $\mu_1(P)$, $\mu_2(P)$, $\mu_3^\pm(P)$ and $\mu_4(P)$ in the $P\mu$ -plane.

and

$$\begin{aligned}\mu_1(P) &= \frac{1}{2} - \frac{1}{3P}, \\ \mu_2(P) &= \begin{cases} \frac{1}{72} (4 + P^2 + 4P^4 - \sqrt{16P^8 + 8P^6 - 543P^4 + 1160P^2 + 16}) & \text{if } P \leq P_*, \\ \frac{P(3P+2)}{6(P^2-2)} & \text{if } P_* \leq P, \end{cases} \\ \mu_3^\pm(P) &= \frac{P^2(3P^4 - 12P^2 + 14) \pm P^2\sqrt{P^8 - 16P^6 + 84P^4 - 176P^2 + 132}}{4(P^2-1)(P^2-2)^2} \\ \mu_4(P) &= \frac{3P^4 - 4P^2 - 2 + \sqrt{P^8 - 12P^4 + 16P^2 + 4}}{4P^2(P^2-1)},\end{aligned}$$

where $P_* \approx 2.88965$ is the unique root of the polynomial

$$U(P) = 6P^4 - P^3 - 38P^2 - 28P + 4$$

on the interval $2 < P < +\infty$ and $P_2 \approx 2.82343$ is the largest root of the polynomial

$$V(P) = P^8 - 16P^6 + 84P^4 - 176P^2 + 132$$

on the positive real axis. Moreover,

$$\frac{1}{3} < \mu_1 < \frac{1}{2} < \mu_2 < \mu_4 < \frac{8}{9},$$

on the interval $2 < P$, and $\mu_2 < \mu_3^- < \mu_3^+ < \mu_4$ on $P_2 < P < P_*$ whereas $\mu_3^- < \mu_2 < \mu_3^+ < \mu_4$ on $P_* < P$.

We see numerically that $p_* \in (0, 1)$ satisfying $P_* = p_* + 1/p_*$ is approximately 0.401984. Also, we have $p_2 \approx 0.415252$ for $p_2 \in (0, 1)$ with $P_2 = p_2 + 1/p_2$.

The behaviour of $\mu_1(P)$, $\mu_2(P)$, $\mu_3^\pm(P)$ and $\mu_4(P)$ can be observed in Figure 1.

The Fekete-Szegő problem was solved by Bhowmik, Ponnusamy and Wirths in [6] for the different but related classes $\mathcal{Co}(\alpha)$ for $1 < \alpha \leq 2$. Here, by definition, $f \in \mathcal{Co}(\alpha)$ if $f \in \mathcal{S}$ with $f(1) = \infty$, if $\mathbb{C} \setminus f(\mathbb{D})$ is convex, and if the opening angle of the image $f(\mathbb{D})$ at ∞ is $\leq \pi\alpha$. It is interesting to observe that the case $\alpha = 2$ of their main theorem in [6] agrees with the limiting case of our Theorem 2 as $p \rightarrow 1^-$ (equivalently, $P \rightarrow 2^+$).

With the special choice $\mu = 0$, we have the following known fact.

Corollary 3. *Let $f(z) = z + a_1z + a_2z^2 + \cdots$ be a function in \mathcal{Co}_p . Then the following sharp inequality holds:*

$$|a_3| \leq P^2 - 1 = p^2 + 1 + \frac{1}{p^2}.$$

Indeed, the above inequality is still valid as long as f is univalent meromorphic on \mathbb{D} with a pole at p (see Jenkins [9]). Avkhadiyev, Pommerenke and Wirths [1] (see also [5]) proved the even stronger result that the variability region of a_3 over $f \in \mathcal{Co}_p$ is given as $|a_3 - P^2 + 2| \leq 1$. (This can also be proved by our method given below.)

Since $\Phi(1) = 1$ by Theorem 2, we get another corollary.

Corollary 4. *Let $0 < p < 1$ and suppose that $f(z) = z + a_1z + a_2z^2 + \cdots$ is a function in \mathcal{Co}_p . Then the following sharp inequality holds:*

$$|a_3 - a_2^2| \leq 1.$$

Recall that $6(a_3 - a_2^2) = S_f(0)$ is the Schwarzian derivative of f evaluated at $z = 0$. The inequality $|a_3 - a_2^2| \leq 1$ is valid for a univalent holomorphic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ on \mathbb{D} (see, for instance, [8, Ex. 1 in p. 70]). Indeed, it is obtained by a simple application of Gronwall's area theorem for the function $1/f(1/w)$. Since the Schwarzian derivative S_f is unchanged under the post-composition with Möbius transformations, the above corollary can also be obtained from this classical result.

In the final section, we will focus on the variability region of $\Lambda_1(f) = a_3 - a_2^2$ over $f \in \mathcal{Co}_p$. Section 2 will be devoted to the proof of Theorem 1. In order to apply Theorem 1 to concave functions, in Section 3, we consider a maximum value problem for a quadratic polynomial over the closed unit disk. The proof of Theorem 2 will be given in Section 4.

2. PROOF OF THEOREM 1

For the proof of Theorem 1, we recall a useful lemma due to Dieudonné (see [8, p. 198] for instance). To clarify the equality case in the lemma, we will give a proof briefly. Throughout this section, it is helpful to use the special automorphism

$$(2.1) \quad T_a(z) = \frac{a - z}{1 - \bar{a}z}$$

of \mathbb{D} for $a \in \mathbb{D}$. This is indeed an analytic involution of \mathbb{D} and interchanges 0 and a . Moreover,

$$T'_a(z) = \frac{|a|^2 - 1}{(1 - \bar{a}z)^2}.$$

In particular,

$$T'_a(0) = |a|^2 - 1 \quad \text{and} \quad T'_a(a) = \frac{1}{|a|^2 - 1}.$$

Lemma 5 (Dieudonné's Lemma). *Let $z_0, w_0 \in \mathbb{D}$ with $|w_0| < |z_0|$. Then the region of values of $w = g'(z_0)$ for holomorphic functions $g : \mathbb{D} \rightarrow \mathbb{D}$ with $g(0) = 0$ and $g(z_0) = w_0$ is given as the closed disk*

$$(2.2) \quad \left| w - \frac{w_0}{z_0} \right| \leq \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |z_0|^2)}.$$

Equality holds if and only if g is a Blaschke product of degree 2 fixing 0.

Proof. The function $h(z) = g(z)/z$ is an analytic endomorphism of \mathbb{D} which sends z_0 to w_0 . Thus $H = T_{w_0} \circ h \circ T_{z_0}$ belongs to \mathcal{B}_0 . The Schwarz lemma now gives $|H'(0)| \leq 1$ which turns out to be equivalent to (2.2) with $w = g'(z_0)$. Moreover, equality holds if and only if $H(z) = \zeta z$ for some $\zeta \in \partial\mathbb{D}$, which means h is an analytic automorphism of \mathbb{D} . \square

In view of the proof, we have a concrete form of g in the equality case:

$$g(z) = zT_{\omega_0}(\zeta T_{z_0}(z))$$

for a constant $\zeta \in \partial\mathbb{D}$, where $\omega_0 = w_0/z_0 \in \mathbb{D}$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. For a function $\varphi \in \mathcal{B}_p$, we consider $\psi = T_p \circ \varphi \circ T_p : \mathbb{D} \rightarrow \mathbb{D}$. Then $\psi \in \mathcal{B}_0$. The Schwarz lemma implies $|\psi(p)| \leq p$. Namely,

$$|T_p(c_0)| = \left| \frac{p - c_0}{1 - p c_0} \right| \leq p,$$

which is equivalent to

$$(2.3) \quad 0 \leq |1 - p c_0|^2 - \left| 1 - \frac{c_0}{p} \right|^2 = \frac{1 - p^4}{p^2} \left[\left(\frac{p}{1 + p^2} \right)^2 - \left| c_0 - \frac{p}{1 + p^2} \right|^2 \right].$$

The range is optimal because the function φ corresponding to $\psi(z) = T_p(c_0)z/p$ belongs to \mathcal{B}_p . Suppose now that $c_0 \in \partial\mathbf{X}_0(\mathcal{B}_p)$. Then, by the above argument, we have $\psi(z) = \zeta z$, where $\zeta = T_p(c_0)/p \in \partial\mathbb{D}$. Thus $\varphi(z) = T_p(\zeta T_p(z))$ is an analytic automorphism of \mathbb{D} fixing p . Hence the first assertion follows.

For the second assertion, we use Dieudonné's lemma. Note that

$$\psi'(p) = T'_p(c_0) \cdot \varphi'(0) \cdot T'_p(p) = \frac{c_1}{(1 - p c_0)^2}.$$

Applying Dieudonné's lemma to the function ψ with the choices $z_0 = p$ and $w_0 = \psi(p) = T_p(c_0)$, we get

$$\left| \frac{c_1}{(1 - p c_0)^2} - \frac{p - c_0}{p(1 - p c_0)} \right| \leq \frac{p^2 - \left| \frac{p - c_0}{1 - p c_0} \right|^2}{p(1 - p^2)}.$$

Here, if $|w_0| = p = |z_0|$, the above inequality (in fact, equality) holds obviously. Note that the above range of c_1 for a fixed c_0 is optimal by Dieudonné's lemma. Using the identity in (2.3), we obtain the first description of the set $\mathbf{X}_1(\mathcal{B}_p)$. The second description of $\mathbf{X}_1(\mathcal{B}_p)$ is obtained by letting $\sigma_0 = P(P^{-1} - c_0) = 1 - P c_0$ and $\sigma_1 = (c_1 - (1 - P c_0 + c_0^2))/(P^{-1} - P|c_0 - P^{-1}|^2) = P(c_1 - P^{-2}(1 + (P^2 - 2)\sigma_0 + \sigma_0^2))/(1 - |\sigma_0|^2)$.

We now prove the final assertion. Suppose that $(c_0, c_1) \in \partial\mathbf{X}_1(\mathcal{B}_p)$ for a function $\varphi(z) = c_0 + c_1 z + \dots$ in \mathcal{B}_p . By part (i), we know that $c_0 \in \partial\mathbf{X}_0(\mathcal{B}_p)$ if and only if φ

is an analytic automorphism of \mathbb{D} fixing p . Thus we may assume that c_0 is an interior point of $\mathbf{X}_0(\mathcal{B}_p)$; namely, $|T_p(c_0)| < p$. Then, by the equality case in Dieudonné's lemma, $\psi = T_p \circ \varphi \circ T_p$ is a Blaschke product of degree 2 fixing 0. Therefore, we have proved the “only if” part. The “if” part is easy to check. \square

3. MAXIMUM VALUE PROBLEM FOR A QUADRATIC POLYNOMIAL

In order to apply Theorem 1 for concave functions, we consider the following problem: What is the value of the quantity

$$(3.1) \quad Y(a, b, c) = \max_{z \in \mathbb{D}} (|a + bz + cz^2| + 1 - |z|^2)$$

for real numbers a, b, c ?

In fact, a more general and symmetric problem was considered in [7]. Let

$$\Omega(A, B, K, L, M) = \max_{u, v \in \mathbb{D}} \{ |A|(1 - |u|^2) + |B|(1 - |v|^2) + |Ku^2 + 2Muv + Lv^2| \}$$

for $A, B, K, L, M \in \mathbb{C}$. When K, L, M are all real numbers, the value of $\Omega(A, B, K, L, M)$ is computed in [7, Theorem 3.1]. By virtue of the maximum modulus principle, one can see that

$$\Omega(1, 0, c, a, b/2) = \max_{u \in \mathbb{D}, v \in \partial\mathbb{D}} \{ (1 - |u|^2) + |cu^2 + buv + av^2| \} = Y(a, b, c).$$

As an immediate consequence of Theorem 3.1 in [7], we obtain the following result. (Note that, under the notation adopted in [7], $\max\{\Phi_1, \Phi_2\} \geq 0$ because of $B = 0$ so that $S \geq |A| + |B| = 1$ in the case (3c) of Theorem 3.1 in [7].)

Proposition 6. *Let $Y(a, b, c)$ be the quantity defined in (3.1) for real numbers a, b, c . When $ac \geq 0$,*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c| & \text{if } |b| \geq 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 - |c|)} & \text{if } |b| < 2(1 - |c|). \end{cases}$$

When $ac < 0$,

$$(3.2) \quad Y(a, b, c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1 - |c|)} & \text{if } -4ac(c^{-2} - 1) \leq b^2 \text{ and } |b| < 2(1 - |c|), \\ 1 + |a| + \frac{b^2}{4(1 + |c|)} & \text{if } b^2 < \min\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\}, \\ R(a, b, c) & \text{otherwise,} \end{cases}$$

where

$$(3.3) \quad R(a, b, c) = \begin{cases} |a| + |b| - |c| & \text{if } |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c| & \text{if } |ab| \leq |c|(|b| - 4|a|), \\ (|c| + |a|)\sqrt{1 - \frac{b^2}{4ac}} & \text{otherwise.} \end{cases}$$

4. PROOF OF THEOREM 2

Let $p \in (0, 1)$ and put $P = p + 1/p$ as before. For a given function $f \in \mathcal{C}o_p$ with expansion (1.4), there is a unique function $\varphi \in \mathcal{B}_p$ with expansion (1.1) such that the representation formula (1.3) holds. A straightforward computation yields

$$a_2 = P - c_0 \quad \text{and} \quad a_3 = P^2 - \frac{1}{3}(c_1 - c_0^2 + 4Pc_0 + 2).$$

For $\mu \in \mathbb{R}$, by substituting the expressions in (1.2), we obtain

$$(4.1) \quad \begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3} [(1 - 3\mu)c_0^2 + 2(3\mu - 2)Pc_0 - c_1 + (3 - \mu)P^2 - 2] \\ &= P^2 - 2 - \mu(P - P^{-1})^2 + (1 - 2\mu(1 - P^{-2}))\sigma_0 - \mu P^{-2}\sigma_0^2 - \frac{(1 - |\sigma_0|^2)\sigma_1}{3P}. \end{aligned}$$

Since σ_1 is an arbitrary point in $\overline{\mathbb{D}}$, we get the sharp inequality

$$|a_3 - \mu a_2^2| \leq \frac{1}{3P} \{ |a + b\sigma_0 + c\sigma_0^2| + 1 - |\sigma_0|^2 \},$$

where

$$a = 3P[P^2 - 2 - \mu(P - P^{-1})^2], \quad b = 3P - 6\mu(P - P^{-1}) \quad \text{and} \quad c = -3\mu P^{-1}.$$

Therefore, in terms of the quantity introduced in the last section, we can express $\Phi(\mu)$ by

$$\Phi(\mu) = \sup_{f \in \mathcal{C}o_p} \Lambda_\mu(f) = \frac{1}{3P} Y(a, b, c).$$

Observe that a changes its sign at $\mu = \mu_a := (P^2 - 2)/(P - P^{-1})^2 > 0$, whereas c changes its sign at $\mu = 0$. It is easy to verify

$$\frac{8}{9} < \mu_a < 1.$$

Furthermore b changes its sign at $\mu = \mu_b := P/2(P - P^{-1}) \in (1/2, 2/3)$.

Case when $\mu \leq 0$: In this case, $a \geq 0, c \geq 0$ and $b \geq 0$. Since $2(1 - |c|) - |b| = 2 - 3P + 6P\mu < 0$, Proposition 6 leads to

$$\Phi(\mu) = \frac{1}{3P} (a + b + c) = (1 - \mu)P - 1.$$

Case when $\mu \geq \mu_a$: In this case, $a \leq 0, b \leq 0$ and $c \leq 0$ and thus $ac \geq 0$. Since $2(1 - |c|) - |b| = 2 + 3P - 6P\mu < 0$ for $\mu \geq \mu_a > 1/2 + 1/3P$, we have by Proposition 6

$$\Phi(\mu) = \frac{1}{3P} (-a - b - c) = (\mu - 1)P + 1.$$

Case when $0 < \mu < \mu_a$: In this case, $a > 0, c < 0$ and thus $ac < 0$. We compute $b^2 + 4ac(c^{-2} - 1) = H(\mu)/\mu$, where H is a quadratic polynomial in μ given by

$$H(\mu) = -36\mu^2 + (4 + P^2 + 4P^4)\mu - 4P^2(P^2 - 2).$$

The roots of $H(\mu)$ are given by

$$\mu_0^\pm = \frac{1}{72} \left(4 + P^2 + 4P^4 \pm \sqrt{16P^8 + 8P^6 - 543P^4 + 1160P^2 + 16} \right).$$

Since $H(2/3) = -2(P^2 - 4)(2P^2 - 5)/3 < 0$, $H(\mu_a) = 9P^4(P^2 - 3)^2(P^2 - 2)/(P^2 - 1)^4 > 0$ and $H(4/3) = 4(P^2 - 4)(P^2 + 11)/3 > 0$, the roots are real and satisfy $2/3 < \mu_0^- < \mu_a < 4/3 < \mu_0^+$. Note that $H(\mu) < 0$ for $\mu \in (-\infty, \mu_0^-) \cup (\mu_0^+, +\infty)$ and that $H(\mu) \geq 0$ for

$\mu \in [\mu_0^-, \mu_0^+]$. Since $2(1 - |c|) - |b| = 2(1 + c) + b = 2 + (3 - 6\mu)P < 2 - P < 0$ for $\mu \geq \mu_0^- (> 2/3)$, the first case in (3.2) does not occur.

We next analyze the condition $b^2 < 4(1 + |c|)^2$, which is equivalent to $|b| < 2(1 + |c|) = 2(1 - c)$ in the present case. We observe that $b < 2(1 - c)$ precisely when $\mu > \mu_1 = 1/2 - 1/3P$ whereas $-b < 2(1 - c)$ precisely when $\mu < \mu'_1 := P(3P + 2)/6(P^2 - 2)$. Note here that $1/3 < \mu_1 < 1/2 < \mu'_1 < 4/3$. Hence, for $\mu \in (0, \mu_a)$, we see that $b^2 < 4(1 + |c|)^2$ if and only if $\mu_1 < \mu < \mu'_1$. Hence, by the second case of (3.2), we obtain

$$\Phi(\mu) = \frac{1}{3P} \left(1 + a + \frac{b^2}{4(1 - c)} \right)$$

for $\mu_1 < \mu < \mu_2 = \min\{\mu_0^-, \mu'_1\}$. Substituting the explicit forms of a, b, c , we obtain the expression in the theorem. Here, keeping $\mu'_1 < 4/3$ in mind, we see that $\mu'_1 > \mu_0^-$ if and only if

$$H(\mu'_1) = -\frac{P(2P - 1)(P^2 - 4)U(P)}{6(P^2 - 2)^2} > 0,$$

where $U(P)$ is the quartic polynomial given in Theorem 2. One can check that the polynomial $U(P)$ has a unique root $P_* \approx 2.88965$ in the interval $2 < P < +\infty$. Thus $\mu_2 = \mu_0^-$ if $2 < P \leq P_*$ and $\mu_2 = \mu'_1$ if $P_* \leq P < +\infty$.

When either $0 < \mu \leq \mu_1$ or $\mu_2 \leq \mu < \mu_a$, we have $Y(a, b, c) = R(a, b, c)$ in (3.2). We shall take a closer look at these cases.

Subcase when $0 < \mu < \mu_1$: Since $\mu_1 < 1/2 < \mu_b$, we have $b > 0$ in this case. We compute $|ab| - |c|(|b| + 4|a|) = ab + c(b + 4a) = 9[2P^2(P^2 - 1)\mu^2 - (3P^4 - 4P^2 - 2)\mu + P^2(P^2 - 2)]$. Note that the above quadratic polynomial in μ is convex and has the axis of symmetry $\mu = (3P^4 - 4P^2 - 2)/4P^2(P^2 - 1) > 1/2 > \mu_1$. Therefore, it is decreasing in $0 < \mu < \mu_1$ and thus

$$\begin{aligned} |ab| - |c|(|b| + 4|a|) &\geq 9[2P^2(P^2 - 1)\mu_1^2 - (3P^4 - 4P^2 - 2)\mu_1 + P^2(P^2 - 2)] \\ &= \frac{9}{2P}(6P^4 - 5P^3 - 12P^2 + 14P - 12) > 0 \end{aligned}$$

for $P > 2$. Hence, by the first case of (3.3) in Proposition 6, we have $\Phi(\mu) = R(a, b, c)/3P = (a + b + c)/3P = (1 - \mu)P^2 - 1$.

Subcase when $\mu_2 < \mu < \mu_a$: First note that $\mu'_1 - \mu_b = P(P + 2)(2P - 1)/6(P^2 - 1)(P^2 - 2) > 0$. We also have $\mu_b < 2/3 < \mu_0^-$. Thus, we observe that $\mu_b < \mu_2 = \min\{\mu_0^-, \mu'_1\}$, which implies that $b < 0$ in this case. Therefore, $|ab| - |c|(|b| + 4|a|) = -ab + c(-b + 4a) = -9P^{-2}F(\mu)$, where

$$F(\mu) = 2(P^2 - 1)(P^2 - 2)^2\mu^2 - P^2(3P^4 - 12P^2 + 14)\mu + P^4(P^2 - 2).$$

The discriminant of $F(\mu)$ is $D = P^4V(P)$, where $V(P)$ is given in Theorem 2. One can see that the polynomial D in P has exactly two roots P_1, P_2 in the interval $2 < P < +\infty$ with $P_1 \approx 2.05313 < P_2 \approx 2.82343$ and that $D \geq 0$ on $P > 2$ if and only if either $2 < P \leq P_1$ or $P_2 \leq P$. The axis of symmetry of $F(\mu)$ is $\mu = \mu_F := P^2(3P^4 - 12P^2 + 14)/4(P^2 - 1)(P^2 - 2)^2$. Since

$$\mu_F - 1 = \frac{P^2(-P^6 + 8P^4 - 18P^2 + 16)}{4(P^2 - 1)(P^2 - 2)^2} > 0 \quad (2 < P \leq 2.2),$$

we have $F(\mu) > F(1) = 2(P^2 - 4) > 0$ for $\mu < 1$ and $2 < P \leq P_1$. Since $F(\mu) > 0$ for all $\mu \in \mathbb{R}$ when $P_1 < P < P_2$, we conclude that $|ab| - |c|(|b| + 4|a|) = -9P^{-2}F(\mu) < 0$ for $\mu < \mu_a (< 1)$ and $2 < P < P_2$.

Solving the equation $F(\mu) = 0$, we write the solutions as

$$\mu_3^\pm = \frac{P^2(3P^4 - 12P^2 + 14) \pm P^2\sqrt{P^8 - 16P^6 + 84P^4 - 176P^2 + 132}}{4(P^2 - 1)(P^2 - 2)^2}$$

for $P \in [P_2, +\infty)$. Note that $F(\mu) > 0$ for $\mu \in (-\infty, \mu_3^-) \cup (\mu_3^+, +\infty)$ and that $F(\mu) \leq 0$ for $\mu \in [\mu_3^-, \mu_3^+]$. As above, we compute

$$\mu_a - \mu_F = \frac{P^2(P^6 - 9P^4 + 22P^2 - 18)}{4(P^2 - 1)^2(P^2 - 2)^2} > 0 \quad (2.5 < P),$$

and

$$F(\mu_a) = \frac{P^4(P^2 - 2)(P^2 - 3)}{(P^2 - 1)^3} > 0,$$

both of which imply that $\mu_3^+ < \mu_a$ for $P_2 \leq P$. On the other hand, for $2 < P$, we see that

$$F(\mu_1') = -\frac{P^2(P - 2)(6P^4 - P^3 - 38P^2 - 28P + 4)}{18(P^2 - 2)} = -\frac{P^2(P - 2)U(P)}{18(P^2 - 2)} \leq 0$$

if and only if $P_* \leq P$, where P_* is the unique root of $U(P)$ in $2 < P < +\infty$ as was introduced above. Hence, $\mu_3^- \leq \mu_1' = \mu_2 \leq \mu_3^+$ when $P_* \leq P$, and either $\mu_1' < \mu_3^-$ or $\mu_3^+ < \mu_1'$ when $P_2 \leq P < P_*$. In view of the fact that

$$\begin{aligned} (\mu_3^- - \mu_1') \Big|_{P=P_2} &= \frac{P_2^2(3P_2^4 - 12P_2^2 + 14)}{4(P_2^2 - 1)(P_2^2 - 2)^2} - \frac{P_2(3P_2 + 2)}{6(P_2^2 - 2)} \\ &= \frac{P_2(3P_2^5 - 4P_2^4 - 18P_2^3 + 12P_2^2 + 30P_2 - 8)}{12(P_2^2 - 1)(P_2^2 - 2)^2} \approx 0.049 > 0, \end{aligned}$$

we can conclude, by continuity, that $\mu_2 = \mu_1' < \mu_3^-$ for $P_2 \leq P < P_*$. (In particular, we see that $\mu_0 = \mu_1' = \mu_3^-$ at $P = P_*$. Look around the point $(P_*, \mu_2(P_*))$ in Figure 1. We wonder if this is just an incidence.)

Similarly, we have $|c|(|b| - 4|a|) - |ab| = -c(-b - 4a) + ab = 9P^{-1}G(\mu)$, where

$$G(\mu) = 2P^2(P^2 - 1)\mu^2 - (3P^4 - 4P^2 - 2)\mu + P^2(P^2 - 2).$$

Solving the equation $G(\mu) = 0$, we write the solutions as

$$\mu_4^\pm = \frac{3P^4 - 4P^2 - 2 \pm \sqrt{P^8 - 12P^4 + 16P^2 + 4}}{4P^2(P^2 - 1)}, \quad 2 < P.$$

Here, we note that $P^8 - 12P^4 + 16P^2 + 4 = (P^4 - 6)^2 + 16P^2 - 32 > 132$ for $2 < P$. We now compute $G(\mu_a) = P^2(P^2 - 2)(P^2 - 3)/(P^2 - 1)^3 > 0$. Since the axis $\mu = \mu_G := (3P^4 - 4P^2 - 2)/4P^2(P^2 - 1)$ of $G(\mu)$ satisfies $\mu_G < 3/4 < \mu_a$, we have $\mu_4^+ < \mu_a$. On the other hand, since

$$\mu_4^- - \frac{1}{2} = \frac{-P^4 - 2P^2 - 2 - \sqrt{P^8 - 12P^4 + 16P^2 + 4}}{4P^2(P^2 - 1)} < 0,$$

we get $\mu_4^- < 1/2 < \mu_2$ for $2 < P$. We now show that $\mu_0^- < \mu_4^+$ for $2 < P$, from which the inequality $\mu_2 < \mu_4^+$ will follow. Since $16P^8 + 8P^6 - 543P^4 + 1160P^2 + 16 - (4P^4 - 8P^2 - 8)^2 =$

$3(P^2 - 4)(24P^4 - 85P^2 + 4) > 0$, we have

$$72\mu_0^+ > 4 + P^2 + 4P^4 + \sqrt{(4P^4 - 8P^2 - 8)^2} = 8P^4 - 7P^2 - 4 > 0$$

for $P > 2$. Therefore,

$$\mu_0^- = \frac{P^2(P^2 - 2)}{9\mu_0^+} < \frac{8P^2(P^2 - 2)}{8P^4 - 7P^2 - 4}.$$

On the other hand, since $P^8 - 12P^4 + 16P^2 + 4 = (P^4 - 6)^2 + 16(P^2 - 2) > (P^4 - 6)^2$, we obtain

$$\mu_4^+ > \frac{3P^4 - 4P^2 - 2 + (P^4 - 6)}{4P^2(P^2 - 1)} = \frac{(P^2 + 1)(P^2 - 2)}{P^2(P^2 - 1)}.$$

Because

$$\frac{(P^2 + 1)(P^2 - 2)}{P^2(P^2 - 1)} - \frac{8P^2(P^2 - 2)}{8P^4 - 7P^2 - 4} = \frac{(P^2 - 2)(9P^4 - 11P^2 - 4)}{P^2(P^2 - 1)(8P^4 - 7P^2 - 4)} > 0$$

for $P > 2$, the inequality $\mu_0^- < \mu_4^+$ follows as required.

We now summarize the above observations as follows. Let $D = \{(P, \mu) : 2 < P, \mu_2(P) < \mu < \mu_a(P)\}$. Here, we write μ_2 etc. as functions of P . We divide D into three parts D_1, D_2, D_3 according as the first, second, third case occurs in (3.3), respectively. Then, $D_1 = \{(P, \mu) : P_2 \leq P < P_*, \mu_3^-(P) \leq \mu \leq \mu_3^+(P)\} \cup \{(P, \mu) : P_* \leq P, \mu_2(P) < \mu \leq \mu_3^+(P)\}$ and $D_2 = \{(P, \mu) : \mu_4^+(P) \leq \mu < \mu_a(P)\}$. Since D_1 and D_2 are disjoint, we have necessarily that $\mu_3^+ < \mu_4^+$ for $P_2 \leq P$. Note here that $\Phi(\mu) = (a - b + c)/3P = P^2 - 3 - \mu(P^2 - 2)^2/P^2$ for $(P, \mu) \in D_1$, that $\Phi(\mu) = (-a - b - c)/3P = (\mu - 1)P^2 + 1$ for $(P, \mu) \in D_2$ and that

$$\begin{aligned} \Phi(\mu) &= \frac{(a - c)}{3P} \sqrt{1 - \frac{b^2}{4ac}} \\ &= (1 - \mu)P(P^2 - 2) \sqrt{\frac{P^2 - 4\mu}{4\mu\{(1 - \mu)(P^2 - 1)^2 - 1\}}} \end{aligned}$$

for $(P, \mu) \in D_3$.

Finally, setting $\mu_4 = \mu_4^+$ for simplicity, we complete the proof of Theorem 2.

5. VARIABILITY REGION OF $a_3 - a_2^2$

We first note that the class $\mathcal{C}o_p$ is not rotationally invariant for $0 < p < 1$ due to the presence of a pole at p . It is therefore more natural to consider the variability region of the Fekete-Szegő functional Λ_μ over $\mathcal{C}o_p$ rather than its modulus only. The present section will be devoted to the study of the variability region of $\Lambda_1(f) = a_3 - a_2^2$ because of its importance. Let

$$W_p = \{\Lambda_1(f) : f \in \mathcal{C}o_p\}$$

for $0 < p < 1$.

In the following, we fix $p \in (0, 1)$ and put $P = p + 1/p > 2$. Let

$$f_\zeta(z) = \frac{z - T_p(p\zeta)z^2}{(1 - z/p)(1 + pz)} = \sum_{n=1}^{\infty} \frac{1 - p^{2n}\zeta}{p^{n-1}(1 - p^2\zeta)} z^n = \sum_{n=1}^{\infty} A_n(\zeta) z^n$$

for $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$. Here, T_p is defined in (2.1). One can check that f_ζ belongs to $\mathcal{C}o_p$ and corresponds to $\varphi(z) = T_p(\zeta T_p(z))$ through (1.3). As Avkhadiev and Wirths [5] pointed out, the function f_ζ with $|\zeta| = 1$ is extremal in important problems for the class $\mathcal{C}o_p$.

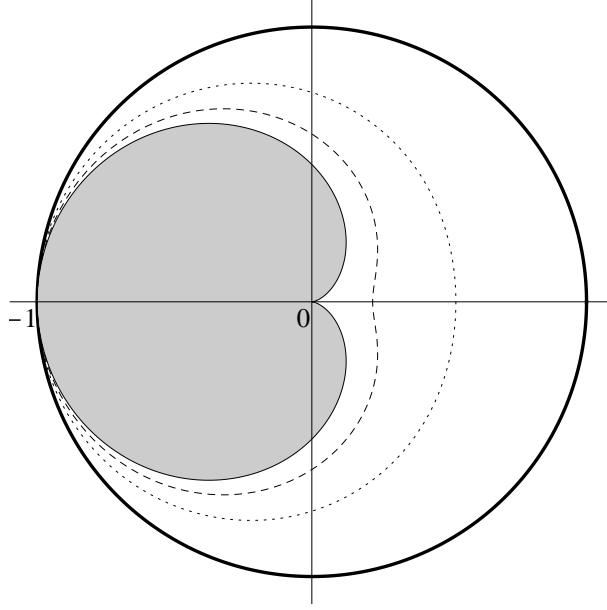


FIGURE 2. A couple of Ω_p 's (the inside of dotted and dashed curves), the intersection cardioid and the unit disk

Indeed, they proved that the closed disk $A_n(\overline{\mathbb{D}})$ is the variability region of the coefficient functional $a_n(f)$ for $f \in \mathcal{C}o_p$ (see also [11]). We now compute

$$\Lambda_1(f_\zeta) = A_3(\zeta) - A_2(\zeta)^2 = -\frac{(1-p^2)^2\zeta}{(1-p^2\zeta)^2} = -(P^2 - 4)K(p^2\zeta),$$

where $K(z) = z/(1-z)^2$ is the Koebe function. One might expect that the image

$$\Omega_p = \{-(P^2 - 4)K(p^2\bar{z}) : |z| \leq 1\}$$

would coincide the variability region W_p . By accident, the form of $A_3 - A_2^2$ is same as the second Hankel determinant $a_2a_4 - a_3^2$ of order 2 for f_ζ ; namely,

$$A_2(\zeta)A_4(\zeta) - A_3(\zeta)^2 = -\frac{(1-p^2)^2\zeta}{(1-p^2\zeta)^2} = A_3(\zeta) - A_2(\zeta)^2.$$

The authors investigated in [11] the set Ω_p in the context of the second Hankel determinant and found that $\Omega_p \subset \Omega_q$ for $0 < q < p < 1$ and that

$$\bigcup_{0 < p < 1} \Omega_p = \mathbb{D} \cup \{-1\} \quad \text{and} \quad \bigcap_{0 < p < 1} \Omega_p = \{-(1+z)^2/4 : |z| \leq 1\}.$$

Note that $\{-(1+z)^2/4 : |z| \leq 1\}$ is a closed Jordan domain bounded by a cardioid (see Figure 2). We also observed in [11] that the variability region of $a_2a_4 - a_3^2$ for $\mathcal{C}o_p$ is properly larger than Ω_p . In the case of $a_3 - a_2^2$, rather surprisingly, the expected result partially holds and a phase transition occurs.

Theorem 7. *Let $0 < p < 1$. The variability region W_p of $a_3 - a_2^2$ for $\mathcal{C}o_p$ satisfies $\Omega_p \subset W_p \subset \overline{\mathbb{D}}$. Moreover, $W_p = \Omega_p$ for $0 < p \leq p_0$ and $W_p \neq \Omega_p$ for $p_0 < p < 1$, where*

$$p_0 = \frac{1 + \sqrt{37} - \sqrt{2(1 + \sqrt{37})}}{6} \approx 0.553175.$$

Proof. Letting $\sigma = T_{p^2}(\zeta)$, we have the representation

$$A_3(\zeta) - A_2(\zeta)^2 = -\frac{p^2}{(1+p^2)^2} [1 - (p^2 + p^{-2})\sigma + \sigma^2] = -P^{-2}h(\sigma),$$

where

$$h(\sigma) = 1 - t\sigma + \sigma^2, \quad t = P^2 - 2 > 2.$$

Hence, $\Omega_p = -P^{-2}h(\overline{\mathbb{D}})$. One can easily check that h is univalent on $\overline{\mathbb{D}}$. Let Δ_r be the image of the closed disk $|z| \leq r$ under the mapping h for $0 \leq r \leq 1$. For $\zeta, \omega \in \partial\mathbb{D}$, the sharp inequality

$$|h(\zeta) - h(r\omega)| = |\zeta - r\omega||\zeta + r\omega - t| \geq (1-r)(t-1-r) = h(r) - h(1),$$

holds. Hence, the Euclidean distance δ_r between $\partial\Delta_r$ and $\partial\Delta_1$ is given as $(1-r)(t-1-r) = (1-r)(P^2-3-r)$ for $0 \leq r \leq 1$. Note that if $|w - h(\sigma)| \leq \delta_r$ for some $\sigma \in \mathbb{C}$ with $|\sigma| = r$, then $w \in \Delta_1$.

Letting $\mu = 1$ in (4.1), we obtain the following representation of $\Lambda_1(f)$ for $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{C}o_p$:

$$\begin{aligned} a_3 - a_2^2 &= -P^{-2} + (2P^{-2} - 1)\sigma_0 - P^{-2}\sigma_0^2 - \frac{(1 - |\sigma_0|^2)\sigma_1}{3P} \\ &= -P^{-2} [h(-\sigma_0) + (1 - |\sigma_0|^2)\sigma_1 P/3] \end{aligned}$$

for some $\sigma_0, \sigma_1 \in \overline{\mathbb{D}}$. Put $r = |\sigma_0|$. Then $h(-\sigma_0) \in \partial\Delta_r$. If $(1-r^2)P/3 \leq \delta_r$, we have $a_3 - a_2^2 \in -P^{-2}\Delta_1 = \Omega_p$. Since

$$\delta_r - (1-r^2)P/3 = \frac{1-r}{3} [3P^2 - (1+r)P - 9 - 3r] \geq \frac{1-r}{3} [3P^2 - 2P - 12],$$

we have $(1-r^2)P/3 \leq \delta_r$ for $P \geq P_0 := (1 + \sqrt{37})/3 \approx 2.36092$, which is the larger root of the polynomial $3P^2 - 2P - 12$. Note that p_0 is determined by $P_0 = p_0 + 1/p_0$. Thus we have shown that $W_p \subset \Omega_p$ for $0 < p \leq p_0$.

We next assume that $2 < P < P_0$. Since $3P^2 - 2P - 12 < 0$, we can find an $r \in (0, 1)$ such that $h(r) - h(1) - (1-r^2)P/3 = (1-r)[3P^2 - (1+r)P - 9 - 3r]/3 < 0$. We choose $\sigma_0 = -r$ and $\sigma_1 = 1$. Then there is a function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in $\mathcal{C}o_p$ satisfying (4.1) with $\mu = 1$:

$$a_3 - a_2^2 = -P^{-2} [h(r) + (1-r^2)P/3].$$

Therefore, we get

$$a_3 - a_2^2 = -P^{-2} [h(1) + \{h(r) - h(1) + (1-r^2)P/3\}] > -P^{-2}h(1) = 1 - 4P^{-2},$$

which implies that $a_3 - a_2^2 \in W_p \setminus \Omega_p$ because $\Omega_p \cap \mathbb{R} = [-1, 1 - 4P^{-2}]$. \square

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